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Calibrated complex structures on the generalized tangent bundle of a Riemannian manifold

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Abstract

We study calibrated complex structures on the generalized tangent bundle of a Riemannian manifold M and their relationship to the Riemannian geometry of M. In particular we introduce a concept of integrability of such structures and we prove that integrability conditions are strictly related to the existence of certain Codazzi tensors on M.

MSC: 53C15; 53C21; 53C42

Keywords: Generalized Geometry; Symplectic vector bundles; Calibrated complex structures

1. Introduction

Let M be a smooth manifold, let TM be the tangent bundle and let T^*M be the cotangent bundle; let us consider the generalized tangent bundle of $M : E = TM \oplus T^*M$; E is, in a natural way, a symplectic vector bundle over M. We are interested in to study calibrated complex structures on E; we find that they are strictly related to the Riemannian geometry of M, then this gives a method of translate problems from TM to $TM \oplus T^*M$ and, in this sense, we are in the context of "Generalized Geometry".

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We recall that the motivation of generalized geometry is to translate problems from *TM* to $TM \oplus T^*M$ in order to work in a more general context, were it is possible to study different problems from the same point of view. Examples are given by Dirac structures and generalized complex structures.

Dirac structures were introduced by Weinstein and Courant in 1990 in order to unify Poisson and pre-symplectic manifolds [2]; generalized complex structures were introduced by Hitchin in [6], and further investigated by Gualtieri in [5], in order to unify symplectic and complex geometry.

In this paper, after recalling the fundamental properties of the geometry of E (Section 2), we introduce calibrated complex structures on this bundle and we prove that they correspond to Riemannian metrics on M, g, with fixed g-symmetric operators on *TM* (Section 3). Then, using a natural bracket on E, defined by the Levi Civita connection of (M, g), we introduce the concept of integrability and we compute explicitly integrability conditions. We find that integrability is strictly related to the Codazzi's tensors theory on (M, g). Precisely any g-symmetric $(1, 1)$ – tensor, H, on M such that H and H^2 are Codazzi tensors, defines an integrable calibrated complex structure on the generalized tangent bundle of M (Section 4). Moreover, in the case of $\mathbb{R}^n$, with the canonical flat Riemannian metric, we are able to completely characterize the integrable calibrated complex structures of E. Other interesting examples come from hypersurfaces isometrically embedded in spaces of constant Riemannian sectional curvature, as, in this case, the second fundamental form is a Codazzi tensor (Section 5). Finally (Section 6), given a calibrated complex structure J on E, we associate to J two Lagrangian subbundles, L_1, L_2; we prove that J is integrable if and only if the sets of smooth sections of L_1 and L_2 are closed under the bracket on E defined by the Riemannian metric on M defined by J. In particular this remark insert our study in the context of Lie algebroids [8].

2. The geometry of $TM \oplus T^*M$

Let M be a smooth manifold of real dimension n, in this section we introduce the basic properties of the geometry of $TM \oplus T^*M$ and some concept of generalized geometry from [2,6].

Let $E = TM \oplus T^*M$, sections of E are elements $X + \xi \in C^\infty(E)$, where $X \in C^\infty(TM)$ is a vector field and $\xi \in C^\infty(T^*M)$ is a $1-$ form. E is equipped with a natural symplectic structure:

$$(X + \xi, Y + \eta) := -\frac{1}{2}(\xi(Y) - \eta(X)) \tag{1}$$

and a natural indefinite metric:

$$\langle X + \xi, Y + \eta\rangle := -\frac{1}{2}(\xi(Y) + \eta(X)) \tag{2}$$

$\langle\, ,\rangle$ is non degenerate and of signature (n, n).

The Courant bracket is defined on sections of E by

$$[X + \xi, Y + \eta]_c := [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi + \frac{1}{2}\mathrm{d}(\xi(Y) - \eta(X)). \tag{3}$$

where $\mathcal{L}$ denotes Lie derivative d is the differential operator and [,] denotes the Lie bracket of vector fields over M.

Now we recall some facts from [2,6].

A Dirac structure on M is a maximal isotropic subbundle L of E (with respect to $\langle\,,\rangle$) such that the set of sections of L, $C^\infty(L)$, is closed under the Courant bracket.

Examples of Dirac structures are given by symplectic manifolds (M, ω), namely $L = \text{graph}(\omega)$ satisfies the condition of Dirac structure.

A generalized complex structure on a real $2n$-dimensional manifold M is a maximal isotropic subbundle L of $E \otimes \mathbb{C}$ such that $L \cap \overline{L} = \{O\}$ and $C^\infty(L)$ is closed under the Courant bracket.

Equivalently, a generalized complex structure can be defined as a complex structure J on E which is $\langle\,,\rangle$ – orthogonal and such that the generalized Nijenhuis tensor of J defined by the Courant bracket is 0. This integrability condition means that the $\pm i$ – eigenbundles L, $\overline{L}$ of J must be closed under the Courant bracket.

Examples of generalized complex structures are given by symplectic manifolds (M, ω), where, in block matrix form is

$$J = \begin{pmatrix} O & -\omega^{-1} \\ \omega & O \end{pmatrix} \tag{4}$$

and by complex manifolds (M, J), where

$$J = \begin{pmatrix} J & O \\ O & -J^* \end{pmatrix} \tag{5}$$

with $J^*(\xi)(Y) := \xi(JY)$.

3. Calibrated complex structures on $TM \oplus T^*M$

Starting from the point of view of generalized geometry introduced at the end of previous section we study complex structures J on E which are (,) – calibrated, that is, such that, for all $\sigma, \tau, \nu \in C^\infty(E)$, $\nu \neq 0$, the following conditions hold:

$$(J\sigma, J\tau) = (\sigma, \tau) \tag{6}$$

$$(\nu, J\nu) > 0. \tag{7}$$

Let g be a Riemannian metric on M, g defines, in a natural way, a complex structure J_g on E by

$$J_g(X + \xi) := -g^{-1}(\xi) + g(X) \tag{8}$$

where $g : TM \to T^*M$ is the bemolle musical isomorphism defined by

$$g(X)(Y) := g(X, Y). \tag{9}$$

We have immediately the following:

Lemma 1. *J_g is (,) – calibrated.*

On the other hand, given a complex structure J on E, (,) – calibrated, we can define a Riemannian metric on M by

$$g(X, Y) := 2(X, J(Y)). \tag{10}$$

We are interested in to investigate relationship between Riemannian metrics on M and (,) – calibrated complex structures on E.

Let $J : E \to E$ be a (,) – calibrated complex structure, J can be written in block matrix form as

$$J = \begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix} \tag{11}$$

where $J_1 : TM \to TM$, $J_2 : T^*M \to TM$, $J_3 : TM \to T^*M$, $J_4 : T^*M \to T^*M$.

Written (,) in the block matrix form $\begin{pmatrix} O & I \\ -I & O \end{pmatrix}$ and written down the conditions:

$$\begin{cases} J^2 = -I \\ (J, J) = (\, , \,), \\ (\, , J) > 0 \end{cases} \tag{12}$$

a direct computation gives the following:

Lemma 2. $J = \begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix}$ *is (,) – calibrated if and only if the following conditions hold:*

$$\begin{cases} J_3 = J_3^* \\ J_3 > 0 \\ J_4 = -J_1^* \\ J_3 J_1 = J_1^* J_3 \\ J_2 = -J_3^{-1}(I + (J_1^*)^2) \\ \begin{pmatrix} J_3 & J_4 \\ -J_1 & -J_2 \end{pmatrix} > 0. \end{cases} \tag{13}$$

Thus a (,) – calibrated complex structure J on E can be written in the block matrix form:

$$\begin{pmatrix} H & -g^{-1}(I + (H^*)^2) \\ g & -H^* \end{pmatrix} \tag{14}$$

where $H : TM \to TM$ is a g-symmetric operator, $H^* : T^*M \to T^*M$ is the dual operator of H defined by $H^*(\xi)(X) := \xi(H(X))$, and $g : TM \to T^*M$ is the bemolle musical isomorphism of the Riemannian metric g on M defined by $g(X, Y) := 2(X, JY)$.

Let H be a matrix of type $n \times n$, we form the $2n \times 2n$ matrix $\begin{pmatrix} O & H \\ O & O \end{pmatrix}$, then is $\exp(H) = \begin{pmatrix} I & H \\ O & I \end{pmatrix}$.

From Lemma 2 we get the following:

Proposition 3. *J is a $(\ ,\ )$ – calibrated complex structure on E if and only if there exist a Riemannian metric g on M and a g-symmetric operator H on TM such that $J = \exp(Hg^{-1})J_g \exp(-Hg^{-1})$.*

Proof. J_g has the block matrix form $\begin{pmatrix} O & -g^{-1} \\ g & O \end{pmatrix}$ and

$$\begin{pmatrix} H & -g^{-1}(I + (H^*)^2) \\ g & -H^* \end{pmatrix} = \begin{pmatrix} I & Hg^{-1} \\ O & I \end{pmatrix} \begin{pmatrix} O & -g^{-1} \\ g & O \end{pmatrix} \begin{pmatrix} I & -Hg^{-1} \\ O & I \end{pmatrix} \quad (15)$$

then we have the statement. □

Previous results can be restated by the following:

Proposition 4. *There is a 1–1 correspondence between Riemannian metrics on M and classes of $(\ ,\ )$ – calibrated structures on E, described by*

$$g \leftrightsquigarrow [J^{(g)}] = \{J : E \to E | J = \exp(Hg^{-1})J_g \exp(-Hg^{-1})\}. \quad (16)$$

Equivalently:

Proposition 5. *There is a 1–1 correspondence between $(\ ,\ )$ – calibrated structures on E and Riemannian metrics g on M with a fixed g-symmetric operator, H, on TM:*

$$J \leftrightsquigarrow (g, H). \quad (17)$$

4. Integrability

In this section, we fix a Riemannian metric g on M then, by using the Levi Civita connection associated to g, we introduce a bracket on sections of E and we define the concept of integrability of complex structures of E.

Let (M, g) be a Riemannian manifold and let ∇ be the Levi Civita connection, ∇ defines a bracket on E in the following way:

$$[X + \xi, Y + \eta]_\nabla := [X, Y] + \nabla_X \eta - \nabla_Y \xi \quad (18)$$

where $[\ ,\ ]$ is the Lie bracket of vector fields on M.

We have the following:

Lemma 6. $[\ ,\ ]_\nabla$ *satisfies the following properties: for all* $X, Y \in C^\infty(TM)$, *for all* $\xi, \eta \in C^\infty(T^*M)$ *and for all* $f \in C^\infty(M)$

1. $[X+\xi, Y+\eta]_\nabla = -[Y+\eta, X+\xi]_\nabla$,
2. $[f(X+\xi), Y+\eta]_\nabla = f[X+\xi, Y+\eta]_\nabla - Y(f)(X+\xi)$, *moreover*
3. *Jacobi identity holds for* $[\ ,\ ]_\nabla$ *if and only if the curvature,* R, *of* ∇ *vanishes.*

Proof.

1. is evident;
2. $[f(X+\xi), Y+\eta]_\nabla = [fX, Y] + \nabla_{fX}\eta - \nabla_Y f\xi = f[X,Y] - Y(f)X + f\nabla_X\eta - Y(f)\xi - f\nabla_Y\xi = f\{[X,Y] + \nabla_X\eta - \nabla_Y\xi\} - Y(f)(X+\xi)$;
3. $[[X+\xi, Y+\eta]_\nabla, Z+\zeta]_\nabla + [[Y+\eta, Z+\zeta]_\nabla, X+\xi]_\nabla + [[Z+\zeta, X+\xi]_\nabla, Y+\eta]_\nabla = [[X,Y],Z] + \nabla_{[X,Y]}\zeta - \nabla_Z\nabla_X\eta + \nabla_Z\nabla_Y\xi + [[Y,Z],X] + \nabla_{[Y,Z]}\xi - \nabla_X\nabla_Y\zeta + \nabla_X\nabla_Z\eta + [[Z,X],Y] + \nabla_{[Z,X]}\eta - \nabla_Y\nabla_Z\xi + \nabla_Y\nabla_X\zeta = -R(X,Y)\zeta - R(Z,X)\eta - R(Y,Z)\xi = -\{\zeta(R(X,Y)\cdot) + \eta(R(Z,X)\cdot) + \xi(R(Y,Z)\cdot)\}$. □

Now remember that given a complex structure J on E the $\pm i$ – eigenbundles of J are subbundles of $E \otimes \mathbb{C}$ and the projection operators, P_+, P_-, are defined by

$$P_\pm := \frac{1}{2}(I \mp iJ). \tag{19}$$

We pose the following:

Definition 7. J is integrable if and only if its eigenbundles are involutive with respect to $[\ ,\ ]_\nabla$, that is if and only if for all $\sigma, \tau \in C^\infty(E)$ we have:

$$P_\mp[P_\pm(\sigma), P_\pm(\tau)]_\nabla = 0. \tag{20}$$

We have:

Lemma 8. *Let* J *be a complex structure on* E *and let*

$$N(J) : C^\infty(E) \times C^\infty(E) \to C^\infty(E) \tag{21}$$

defined by

$$N(J)(\sigma,\tau) := [J\sigma, J\tau]_\nabla - J[J\sigma,\tau]_\nabla - J[\sigma, J\tau]_\nabla - [\sigma,\tau]_\nabla \tag{22}$$

for all $\sigma, \tau \in C^\infty(E)$; $N(J)$ *is an antisymmetric tensor which is called generalized Nijenhuis tensor.*

Proof. Let $\sigma = X + \xi$, $\tau = Y + \eta \in C^\infty(E)$ and let $f \in C^\infty(M)$, denoted $\rho : E \to TM$ the map defined by $\rho(X+\xi) = X$, we have:

$$\begin{aligned} N(J)(f\sigma,\tau) &:= [Jf\sigma, J\tau]_\nabla - J[Jf\sigma,\tau]_\nabla - J[f\sigma, J\tau]_\nabla - [f\sigma,\tau]_\nabla \\ &= fN(J)(\sigma,\tau) - \rho(J\tau)(f)J\sigma + J(\rho(\tau))(f)J\sigma + J(\rho(J\tau))(f)\sigma \\ &\quad + \rho(\tau)(f)\sigma = fN(J)(\sigma,\tau); \end{aligned}$$

the antisymmetric property follows from the antisymmetry of the bracket. □

The following holds:

Lemma 9. $P_\mp[P_\pm(\sigma), P_\pm(\tau)]_\nabla = -\frac{1}{4} P_\mp(N(J)(\sigma, \tau))$.

Proof. We have:

$$\begin{aligned} P_\mp[P_\pm(\sigma), P_\pm(\tau)]_\nabla &= \frac{1}{2}\{[P_\pm(\sigma), P_\pm(\tau)]_\nabla \pm iJ[P_\pm(\sigma), P_\pm(\tau)]_\nabla\} \\ &= \frac{1}{8}\{[\sigma \mp iJ\sigma, \tau \mp iJ\tau]_\nabla \pm iJ[\sigma \mp iJ\sigma, \tau \mp iJ\tau]_\nabla\} \\ &= \frac{1}{8}\{[\sigma, \tau]_\nabla \mp i[J\sigma, \tau]_\nabla \mp i[\sigma, J\tau]_\nabla - [J\sigma, J\tau]_\nabla\} \\ &\quad \pm \frac{1}{8} i\{J[\sigma, \tau]_\nabla \mp iJ[J\sigma, \tau]_\nabla \mp iJ[\sigma, J\tau]_\nabla - J[J\sigma, J\tau]_\nabla\} \\ &= -\frac{1}{8}(N(J)(\sigma, \tau) \pm iJN(J)(\sigma, \tau)) \\ &= -\frac{1}{4} P_\mp(N(J)(\sigma, \tau)). \qquad \square \end{aligned}$$

Then we get:

Corollary 10. *J is integrable if and only if $N(J) = 0$.*

Before to investigate integrability conditions we recall the following definition [1]:

Definition 11. A symmetric two tensor field h on a Riemannian manifold (M, g) will be called a Codazzi tensor if h satisfies the Codazzi equation:

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z) \tag{23}$$

for all X, Y, Z tangent vectors.

In the following we will identify the endomorphisms H and H^2 of TM with the corresponding $(2, 0)$ – tensors h and h^2 defined, respectively, by

$$h(X, Y) = g(H(X), Y) \tag{24}$$

and

$$h^2(X, Y) = g(H^2(X), Y). \tag{25}$$

In particular we say that H and H^2 are Codazzi tensors if and only if h and h^2 are Codazzi tensors.

We have immediately the following:

Lemma 12. *H is a Codazzi tensor if and only if*:

$$(\nabla_X H)Y - (\nabla_Y H)X = 0. \tag{26}$$

Let $J \in [J^{(g)}]$, from (14), we have:

$$\boxed{J(X+\xi) = H(X) - g^{-1}(I + (H^*)^2)\xi + g(X) - H^*\xi} \tag{27}$$

for some g-symmetric operator $H : TM \to TM$ and we get:

Proposition 13. *$N(J)(X, Y) \in C^\infty(TM)$ for all $X, Y \in C^\infty(TM)$ if and only if H is a Codazzi tensor.*

Proof. Let $X, Y \in C^\infty(TM)$, denote:

$$N(J)(X, Y) := Z + \zeta$$

where $Z \in C^\infty(TM)$ and $\zeta \in C^\infty(T^*M)$, we have:

$$\begin{aligned} \zeta &= \nabla_{H(X)} g(Y) - \nabla_{H(Y)} g(X) - g[H(X), Y] - H^*\nabla_Y g(X) \pm g[X, H(Y)] \\ &\quad + H^*\nabla_X g(Y) = -g\{\nabla_X H(Y) - \nabla_Y H(X) + H\nabla_Y X - H\nabla_X Y\} \\ &= -g((\nabla_X H)Y - (\nabla_Y H)X) \end{aligned}$$

then $\zeta = 0$ if and only if $(\nabla_X H)Y = (\nabla_Y H)X$ and this gives the statement. □

Moreover:

Proposition 14. *J is integrable if and only if for all $X, Y \in C^\infty(TM)$ the following conditions are satisfied*:

$$\begin{cases} (\nabla_X H)Y - (\nabla_Y H)X = 0 \\ (\nabla_{H(X)} H)Y - (\nabla_{H(Y)} H)X = 0. \end{cases} \tag{28}$$

Proof. Let $X, Y \in C^\infty(TM)$ let us compute:

$$\begin{aligned} N(J)(X, Y) &= (\nabla_{H(X)} H)Y - (\nabla_{H(Y)} H)X - H((\nabla_X H)Y - (\nabla_Y H)X) \pm g((\nabla_X H)Y \\ &\qquad - g(\nabla_Y H)X), \\ N(J)(X, g(Y)) &= -(\nabla_X H)Y + (\nabla_Y H)X + H((\nabla_X H)(H(Y)) + (\nabla_{H^2(Y)} H)X \\ &\qquad - (\nabla_{H(X)} H^2)Y + g((\nabla_X H)H(Y) - (\nabla_{H(X)} H)Y, \\ N(J)(g(X), g(Y)) &= H((\nabla_X H)Y) + (\nabla_X H)H(Y) - H((\nabla_Y H)X) \pm (\nabla_Y H)H(X) \\ &\qquad + (\nabla_{H(H(X))} H)H(Y) - (\nabla_{H(H(Y))} H)H(X) + H(\nabla_{HH(X)})H)Y \\ &\qquad - (\nabla_{H(H(Y))} H)X) + g((\nabla_X H)(Y) + (\nabla_{H^2 X} H)(Y) - (\nabla_Y H)(X) \\ &\qquad - (\nabla_{H^2 Y} H)(X)); \end{aligned}$$

then it is easily seen that $N(J) = 0$ if and only if the conditions in the statement are satisfied. □

Using the identity:

$$(\nabla_X H^2)(Y) = (\nabla_X H)(H(Y)) + H(\nabla_X H)(Y) \tag{29}$$

we can restate Proposition 15 as in the following:

Proposition 15. *J is integrable if and only if for all $X, Y \in C^\infty(TM)$ the following conditions are satisfied*:

$$\boxed{\begin{cases} (\nabla_X H)Y - (\nabla_Y H)X = 0 \\ (\nabla_X H^2)Y - (\nabla_Y H^2)X = 0 \end{cases}.} \tag{30}$$

In particular we get:

Proposition 16. *J is integrable if and only if H an H^2 are Codazzi tensors.*

5. Examples

In this section we will describe some examples.

5.1. *The case of* $\mathbb{R}^n$

Let us consider the case of $\mathbb{R}^n$ with the standard flat Riemannian metric, g_o. It is well known [4] that a symmetric $(2, 0)$ – tensor, h, on $(\mathbb{R}^n, g_o)$ is a Codazzi tensor if and only if there exists $f \in C^\infty(\mathbb{R}^n)$ such that

$$h = \mathrm{Hess}(f). \tag{31}$$

Then, given $f \in C^\infty(\mathbb{R}^n)$, $h = \mathrm{Hess}(f)$ defines an integrable complex structure $J = J_H$ on $T\mathbb{R}^n \oplus T^*\mathbb{R}^n$, as described in (27), if and only if we have $h^2 = \mathrm{Hess}(\phi)$ for some $\phi \in C^\infty(\mathbb{R}^n)$, that is if and only if, for any $i, j = 1, \ldots, n$, it results:

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_k} \frac{\partial^2 f}{\partial x_k \partial x_j}. \tag{32}$$

This remark allows us to write down explicitly a lot of examples of non parallel H that define integrable calibrated complex structures J_H on E.

For sake of simplicity, in the following, we will consider the case $n = 2$ and we will illustrate two examples.

As first example we have:

Proposition 17. *Let*

$$f \begin{pmatrix} x \\ y \end{pmatrix} = (ax + by)^k \tag{33}$$

with $a, b \in \mathbb{R}$, $a^2 + b^2 > 0$, $k \in \mathbb{N}$, $k \geq 3$, *and let* $h = \mathrm{Hess}(f)$, *then the function*

$$\phi\begin{pmatrix} x \\ y \end{pmatrix} = \frac{k^2(k-1)^2(a^2+b^2)}{2(k-2)(2k-3)}(ax+by)^{2k-2} \tag{34}$$

satisfies the condition: $h^2 = \mathrm{Hess}(\phi)$.

Proof. Let x, y be coordinates on $\mathbb{R}^2$, ϕ is a solution of following equations:

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial x^2} = \left(\dfrac{\partial^2 f}{\partial x^2}\right)^2 + \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2 \\ \dfrac{\partial^2 \phi}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial x \partial y}\left(\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}\right) \\ \dfrac{\partial^2 \phi}{\partial y^2} = \left(\dfrac{\partial^2 f}{\partial y^2}\right)^2 + \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2 \end{cases} \tag{35}$$

that is

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial x^2} = (k(k-1))^2 a^2 (a^2+b^2)(ax+by)^{2(k-2)} \\ \dfrac{\partial^2 \phi}{\partial x \partial y} = (k(k-1))^2 ab(a^2+b^2)(ax+by)^{2(k-2)} \\ \dfrac{\partial^2 \phi}{\partial y^2} = (k(k-1))^2 b^2 (a^2+b^2)(ax+by)^{2(k-2)} \end{cases}. \tag{36}$$

Integrating twice each equation and comparing, we get ϕ. □

As second example we have:

Proposition 18. *Let*

$$f\begin{pmatrix} x \\ y \end{pmatrix} = ax^3 + bx^2y + cxy^2 + dy^3 \tag{37}$$

where $a, b, c, d \in \mathbb{R}$, *then* $h = \mathrm{Hess}(f)$ *defines an integrable calibrated complex structure on* $T\mathbb{R}^2 \oplus T^*\mathbb{R}^2$ *if and only if* a, b, c, d *satisfy the following condition*:

$$b^2 + c^2 - 3ac - 3bd = 0. \tag{38}$$

In this case the function:

$$\phi\begin{pmatrix} x \\ y \end{pmatrix} = \left(3a^2 + \frac{b^2}{3}\right)x^4 + 2(b^2+c^2)x^2y^2 + 4b\left(a + \frac{c}{3}\right)x^3y + ac\left(d + \frac{b}{3}\right)y^3x + \left(3d^2 + \frac{c^2}{3}\right)y^4 \tag{39}$$

satisfies the condition $h^2 = \mathrm{Hess}(\phi)$.

Proof. In this case ϕ is a solution of following equations:

$$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} = 4(9a^2+b^2)x^2 + 4(b^2+c^2)y^2 + 8b(3a+c)xy \\ \dfrac{\partial^2\phi}{\partial x\partial y} = 4b(3a+c)x^2 + 4c(b+3d)y^2 + 4(b^2+3db+3ac+c^2)xy \\ \dfrac{\partial^2\phi}{\partial y^2} = 4(9d^2+c^2)y^2 + 4(b^2+c^2)x^2 + 8c(3d+b)xy \end{cases} . \tag{40}$$

Then the compatibility condition:

$$\frac{\partial^3\phi}{\partial x^2\partial y} = \frac{\partial^3\phi}{\partial x\partial y\partial x} \tag{41}$$

gives immediately:

$$b^2 + c^2 - 3ac - 3bd = 0. \tag{42}$$

On the other hand, integrating directly previous equations with this condition, we get ϕ. □

5.2. Submanifolds

Let (M, g) and $(\tilde{M}, \tilde{g})$ be Riemannian manifolds and let $f : M \to \tilde{M}$ be an isometric immersion; let ∇ and $\tilde{\nabla}$ be the Levi Civita connection of g and $\tilde{g}$, respectively, the second fundamental form, h, of the immersion is defined on tangent vector fields X, Y, over M by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y). \tag{43}$$

Let υ be a vector field of the normal bundle $TM^{\perp}$, let $H_\upsilon : TM \to TM$ defined by

$$g(H_\upsilon(X), (Y)) = \tilde{g}(h(X, Y), \upsilon) \tag{44}$$

for all X, Y vector fields of TM, H_υ is a g-symmetric operator on TM called shape operator.

From Codazzi equation, [3,7], it follows that if $\tilde{M}$ has constant Riemannian sectional curvature then H_υ is a Codazzi tensor.

In particular the shape operator for totally geodesic or totally umbilic submanifolds gives examples of integrable calibrated complex structures on E.

The Euclidean sphere $S^n \subset \mathbb{R}^{n+1}$ provides an example and, using results from [9], we get that the shape operator of the Euclidean sphere S^n, isometrically embedded as hypersurface of an elliptic or a hyperbolic space, gives too an integrable complex structure on E.

In the case M is a hypersurface of $\mathbb{R}^{n+1}$, denoted $H = H_\upsilon$, the following holds:

$$\mathrm{Ricci}(X, Y) = (\mathrm{trace}\, H)g(H(X), Y) - g(H(X), H(Y)). \tag{45}$$

In particular, the operator on TM representing the third fundamental form, H^2, is given by

$$H^2 = (\mathrm{trace}\, H)H - S \tag{46}$$

where S is the operator on TM representing the Ricci tensor.

This equation can be used to produce examples of integrable structures on E.

6. Lie algebroids

In this section, we define two Lagrangian subbundles of E, associated to a calibrated complex structure; we prove that in the case the structure is integrable they are Lie algebroids.

Let (M, g) be a Riemannian manifold, let $g : TM \to T^*M$ be the bemolle isomorphism, let H be a g-symmetric operator on TM and let $J = J_H \in [J^{(g)}]$ be the associated calibrated complex structure on E. We define:

$$L_1 = \text{graph}(g + gH) \tag{47}$$

and

$$L_2 = \text{graph}(g + gH^2). \tag{48}$$

We have the following:

Lemma 19. L_1 and L_2 *are Lagrangian subbundles of* E.

Proof. Let σ, τ be sections of L_1, we can write:

$$\sigma = X + g(X) + g(H(X)) \tag{49}$$

and

$$\tau = Y + g(Y) + g(H(Y)) \tag{50}$$

with $X, Y \in C^\infty(TM)$. We have:

$$(\sigma, \tau) = -\frac{1}{2}(g(X) + g(H(X)))(Y) - (g(Y) + g(H(Y))(X)) \tag{51}$$

$$(\sigma, \tau) = -\frac{1}{2}(g(X, Y) + g(H(X), Y) - g(Y, X) - g(H(Y), X)) \tag{52}$$

$$(\sigma, \tau) = 0. \tag{53}$$

Analogously for sections of L_2, namely we have:

$$(X + g(X) + g(H^2(X)), Y + g(Y) + g(H^2(Y))) = 0 \tag{54}$$

and thus the proof is complete. □

Now let ∇ be the Levi Civita connection of g and let $[\ ,\ ]_\nabla$ be the bracket on E defined in Section 4, denote by $C^\infty(L_i)$ the set of smooth sections of L_i, $i = 1, 2$, we have the following:

Proposition 20. *Let* $\sigma, \tau \in C^\infty(L_1)$, *then* $[\sigma, \tau]_\nabla \in C^\infty(L_1)$ *if and only if* H *is a Codazzi tensor.*

Proof. Let us compute:

$$[\sigma, \tau]_\nabla = [X + g(X) + g(H(X)), Y + g(Y) + g(H(Y))]_\nabla$$
$$= [X, Y] + g([X, Y]) + g(H[X, Y]) + (\nabla_X H)(Y) - (\nabla_Y H)(X)$$

then

$$[\sigma, \tau]_\nabla \in C^\infty(L_1) \text{ if ad only if } (\nabla_X H)(Y) - (\nabla_Y H)(X) = 0,$$

which is the statement. □

Analogously for L_2, we get:

Proposition 21. *Let $\sigma, \tau \in C^\infty(L_2)$, then $[\sigma, \tau]_\nabla \in C^\infty(L_2)$ if and only if H^2 is a Codazzi tensor.*

Proof. Repeat previous computation substituting H^2 to H. □

In particular previous results can be restated as in the following:

Proposition 22. *$C^\infty(L_1)$ and $C^\infty(L_2)$ are closed with respect to $[\ ,\ ]_\nabla$ if and only if J is integrable.*

We recall the definition of Lie algebroid:

Definition 23. A *Lie algebroid* on a smooth manifold M is a vector bundle L over M such that: a Lie bracket, $[\ ,\ ]$, is defined on $C^\infty(L)$, a smooth bundle map $\rho : L \to TM$, called *anchor*, is defined and, for all $\sigma, \tau \in C^\infty(L)$, for all $f \in C^\infty(M)$, the following conditions hold:

$$\rho([\sigma, \tau]) = [\rho(\sigma), \rho(\tau)] \tag{55}$$

$$[f\sigma, \tau] = f([\sigma, \tau]) - (\rho(\tau)(f))\sigma. \tag{56}$$

Using previous notations for (M, g), J, L_1, L_2, we have the following:

Proposition 24. *If J is an integrable calibrated complex structure on E then L_1 and L_2 are Lie algebroids.*

Proof. Let $[\ ,\ ] = [\ ,\ ]_\nabla$ and let $\rho : L_i \to TM$, $i = 1, 2$, be the projection defined by

$$\rho(X + g(X) + gH^i(X)) := X, \tag{57}$$

from Lemma 6, it is enough to verify that Jacobi identity holds for

$$[\ ,\ ]_{\nabla | C^\infty(L_i)}.$$

For $\sigma, \tau, \upsilon \in C^\infty(E)$, we denote:

$$\mathrm{Jac}(\sigma, \tau, \upsilon) := [[\sigma, \tau]_\nabla, \upsilon]_\nabla + [[\tau, \upsilon]_\nabla, \sigma]_\nabla + [[\upsilon, \sigma]_\nabla, \tau]_\nabla, \tag{58}$$

for $X, Y, Z \in C^\infty(TM)$ is

$$\mathrm{Jac}(X, Y, Z) := [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \tag{59}$$

$$\mathrm{Bianchi}(X, Y, Z) := R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0; \tag{60}$$

then, for $\sigma, \tau, \upsilon \in C^\infty(L_i)$, we have:

$$\begin{aligned}
\mathrm{Jac}(\sigma, \tau, \upsilon) &= \mathrm{Jac}(X, Y, Z) \pm g(\mathrm{Bianchi}(X, Y, Z) + R(X, Y)H^i(Z) \\
&\quad + R(Y, Z)H^i(X) + R(Z, X)H^i(Y)) \\
&= -g\{R(X, Y)H^i(Z) + R(Y, Z)H^i(X) + R(Z, X)H^i(Y)\} \\
&= -g(\nabla_X((\nabla_Y H^i)(Z) - (\nabla_Z H^i)(Y)) + \nabla_Y((\nabla_Z H^i)(X) - (\nabla_X H^i)(Z)) \\
&\quad + \nabla_Z((\nabla_X H^i)(Y) - (\nabla_Y H^i)(X)) + ((\nabla_X H^i)(\nabla_Y Z - \nabla_Z Y) \\
&\quad - (\nabla_{[Y,Z]} H^i)(X)) + -((\nabla_Y H^i)(\nabla_X Z - \nabla_Z X) - (\nabla_{[X,Z]} H^i)(Y)) \\
&\quad \pm ((\nabla_Z H^i)(\nabla_Y X - \nabla_X Y) - (\nabla_{[Y,X]} H^i)(Z)) \\
&\quad - H(\mathrm{Bianchi}(X, Y, Z))),
\end{aligned}$$

then the statement follows from Proposition 16. □

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